TABLE I
SUMMARY OF MEASURED DATA FOR A C-BAND LANGE COUPLER AND ITS "UNFOLDED" COMPLEMENT

	BANDWIDTH (%)	MEAN POWER SPLIT (dB)	MAX. DEVIATION FROM MEAN POWER SPLIT (dB)	MAX. INSERTION LOSS (dB)	MIN. ISOLATION (dB)	MAX. VSWR	MAX. DEVIATION FROM TRUE QUADRATURE
REGULAR LANGE COUPLER	73	3.43	± .5	.4	18	1.32	8.5°
"UNFOLDED" LANGE COUPLER	68	3.3	± .5	.32	17.5	1.27	± 2°

velop a family of broad-band backward-wave microstrip couplers with coupling values other than 3 dB. The significant practical advantage when designing on 0.025-in ceramic microstrip may be the elimination of ultranarrow gapwidths as required, for example, by Podell's "wiggly"-line backward-wave coupler [3] for coupling values typically under 7 dB. This extension, however, has not been pursued.

ACKNOWLEDGMENT

The authors wish to thank J. Hickox and R. Pereira for their efforts in fabrication and test.

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Dissipation and Scattering Matrices of Lossy Junctions

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***Abstract*—The purpose of this short paper is to construct the dissipation and scattering matrices of lossy junctions in terms of the eigenvalues of the dissipation matrix. This removes the need to rely on inequality relations between the scattering parameters of lossy circulators. The eigenvalues of the dissipation, scattering, and admittance matrices are related. The eigenvalues of the dissipation matrix give the dissipation associated with each possible way of exciting the junction. The ones of the scattering matrix give the reflection coefficients associated with these different excitations. The admittance eigenvalues define in each instance the eigennetworks of the junction. This leads to the definition of the entries of the dissipation matrix in terms of the loaded and unloaded *Q*-factors of the junction eigennetworks. The scattering matrices of a number of lossy 3-port junctions are also constructed directly in terms of the elements of the eigennetworks.**

I. INTRODUCTION

The general relation between the coefficients of the scattering matrix for a lossy symmetrical 3-port circulator has been discussed by a number of authors [1]–[4]. The insertion loss has also been derived in the case of the lumped-element circulator [5]. The most general relation between the scattering coefficients has been given graphically [3] in terms of the dissipation matrix. Inequality relations for semi-ideal circulators in which the insertion loss is not zero and either the isolation or VSWR is ideal have also been discussed [4].

The purpose of this short paper is to directly construct the dissipation and scattering matrices of lossy junctions in terms of the eigenvalues of the dissipation matrix. This removes the need to rely on inequality relations between the scattering parameters of lossy circulators. The scattering matrix of lossy circulators is also constructed directly in terms of the elements of the eigennetworks.

Manuscript received August 18, 1971; revised June 22, 1972.
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The short paper starts by relating the eigenvalues of the dissipation, scattering, and admittance matrices. The eigenvalues of the dissipation matrix give the dissipation associated with each possible way of exciting the junction. The ones of the scattering matrix give the reflection coefficients associated with these different excitations. The admittance eigenvalues define in each instance the eigennetworks of the junction. This leads to the definition of the entries of the dissipation matrix in terms of the loaded and unloaded Q-factors of the junction eigennetworks. In the most usual arrangement, one of the eigenvalues is associated with a nonresonant shunt network that appears as a short circuit at the reference terminals of the junction, and is therefore always unity. The other eigenvalues are the reflection coefficients of resonant shunt networks, and the presence of loss means that the magnitudes of these eigenvalues will depart from unity. The amplitudes of the eigenvalues are, in general, unequal.

The theory is applied to reciprocal 3-port junctions, to 3-port junction circulators, and to semi-ideal 3-port circulators. It may also be applied to the scattering matrices associated with the different stages in the adjustment of the circulators described elsewhere [7], [11], [12].

II. EIGENVALUES OF SCATTERING AND DISSIPATION MATRICES

For a lossy circulator, the dissipation matrix Q must be positive real [3], [6]:

$$Q = I - (S^*)^T(S) \tag{1}$$

where I is a unit matrix and S is the scattering matrix.

In the case of a symmetrical 3-port junction, one has for the S matrix

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{13} & S_{11} & S_{12} \\ S_{12} & S_{13} & S_{11} \end{bmatrix}. \tag{2}$$

The matrix Q is given by [3], [4], [6]

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{13} & q_{11} & q_{12} \\ q_{12} & q_{13} & q_{11} \end{bmatrix} \tag{3}$$

where

$$q_{11} = 1 - |S_{11}|^2 - |S_{12}|^2 - |S_{13}|^2 \tag{4}$$

$$q_{12} = S_{11}S_{12}^* + S_{12}S_{13}^* + S_{13}S_{11}^* \tag{5}$$

$$q_{13} = q_{12}^*. \tag{6}$$

The matrix Q is positive real provided

$$|q_{12}| < q_{11}. \tag{7}$$

The bounds of (7) are given in [3] with q_{11} and $|q_{12}|$ as parameters. This gives the allowable values of the scattering parameters.

In what follows, the above inequality relation between the scattering parameters will be replaced by implicit ones. This is done by directly forming the S- and Q-matrices of the junction in terms of eigenvalues of the dissipation matrix. These eigenvalues represent the dissipation associated with each possible way of exciting the junction. They are related to the loaded and unloaded Q-factors of the junction eigennetworks.

If the scattering and dissipation matrices have common eigenvectors, their eigenvalues may be related by the following theorem

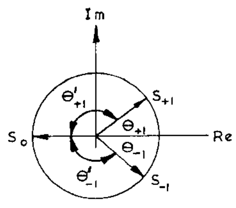

Fig. 1. Scattering-matrix eigenvalues.

[8]: if

$$QU_n = q_n U_n \tag{8}$$

then

$$f(Q)U_n = f(q_n)U_n \tag{9}$$

where U_n is an eigenvector.

Using (1) in conjunction with (9) gives, in the case of the 3-port junction,

$$q_0 = 1 - s_0 s_0^* \tag{10}$$

$$q_{+1} = 1 - s_{+1} s_{+1}^* \tag{11}$$

$$q_{-1} = 1 - s_{-1} s_{-1}^*. \tag{12}$$

Equations (10)–(12) may be used to construct the scattering matrix in terms of the eigenvalues of the dissipation one. In a lossless junction, the amplitudes of the scattering-matrix eigenvalues are unity, while if the junction is lossy their amplitudes will depart from unity. The angles between the eigenvalues that apply to a circulator with $S_{12} = -1$ are shown in Fig. 1.

It is seen from the above that the eigenvalues of the dissipation matrix represent the dissipation associated with each possible way of exciting the junction. These eigenvalues are real quantities that become zero when the corresponding ones of the scattering matrix become unity. The entries of the dissipation matrix are

$$q_{11} = \frac{q_0 + q_{+1} + q_{-1}}{3} \tag{13}$$

$$q_{12} = \frac{q_0 + q_{+1}e^{j2\pi/3} + q_{-1}e^{-j2\pi/3}}{3} \tag{14}$$

$$q_{13} = \frac{q_0 + q_{+1}e^{-j2\pi/3} + q_{-1}e^{j2\pi/3}}{3}. \tag{15}$$

Here, q_{11} represents the total dissipation of the junction and q_{12} is a complex quantity that determines the allowable relation between the scattering parameters. This has been observed in [3].

In what follows the eigenvalues of the Q-matrix will be related to the loaded and unloaded Q-factors of the junction eigennetworks. This does away with the need to rely on the inequality relation given by (7).

The relation between the entries of the scattering matrix is also obtained directly by evaluating this matrix in terms of its eigenvalues. These are given by [8]

$$S_{11} = \frac{s_0 + s_{+1} + s_{-1}}{3} \tag{16}$$

$$S_{12} = \frac{s_0 + s_{+1}e^{j2\pi/3} + s_{-1}e^{-j2\pi/3}}{3} \tag{17}$$

$$S_{13} = \frac{s_0 + s_{1+}e^{-j2\pi/3} + s_{-1}e^{j2\pi/3}}{3}. \tag{18}$$

The scattering eigenvalues are related here to the dissipation ones by (10)–(12).

III. Eigenvalues of Admittance and Dissipation Matrices

The dissipation and scattering eigenvalues can be related to the physical variables of the junction eigennetworks by using the relation between the eigenvalues of the scattering and admittance matrices:

$$s_0 = \frac{1 - y_0}{1 + y_0} \tag{19}$$

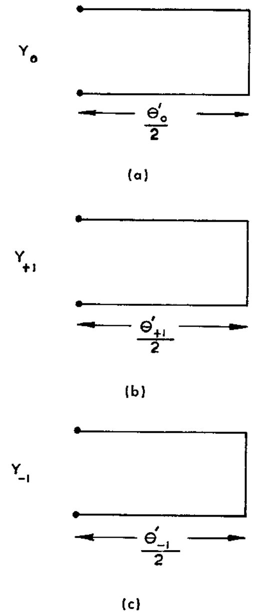

Fig. 2. Distributed eigennetworks.

$$s_{+1} = \frac{1 - y_{+1}}{1 + y_{+1}} \tag{20}$$

$$s_{-1} = \frac{1 - y_{-1}}{1 + y_{-1}}. \tag{21}$$

The admittance eigenvalues are related to the angles that the scattering-matrix eigenvalues make by

$$y_0 = -j \cot \frac{\theta_0'}{2} \tag{22}$$

$$y_{+1} = -j \cot \frac{\theta_{+1}'}{2} = j \tan \frac{\theta_{+1}}{2} \tag{23}$$

$$y_{-1} = -j \cot \frac{\theta_{-1}'}{2} = j \tan \frac{\theta_{-1}}{2}. \tag{24}$$

The primed and unprimed angles are shown in Fig. 1.

The equivalent circuits of the admittance eigenvalues y_{+1} and y_{-1} are short-circuited transmission lines of electrical length $\theta_{+1}'/2$ and $\theta_{-1}'/2$, as shown in Fig. 2. For 3-port circulators for which $S_{12} = -1$, the electrical length θ_0' of the admittance eigenvalue s_0 is zero. One way in which damping may be introduced into the admittance equation is by adding an imaginary part $j\alpha$ to $\theta'/2$.

In the presence of loss, the input admittance of the y_{+1} network becomes

$$y_{+1} = \coth\left(j\alpha_{+1}l + \frac{\theta_{+1}'}{2}\right). \tag{25}$$

The reflection coefficient of this network is

$$s_{+1} = -e^{-2\alpha_{+1}l}\cdot e^{-j\theta'_{+1}} = e^{-2\alpha_{+1}l}\cdot e^{-j\theta_{+1}}. \tag{26}$$

If $2\alpha_{+1}l$ is small, (26) becomes

$$s_{+1} \approx (1 - 2\alpha_{+1}l)e^{-j\theta_{+1}}. \tag{27}$$

Substituting (27) into (11) gives

$$q_{+1} \approx 4\alpha_{+}l. \tag{28}$$

Equation (28) again shows that the dissipation eigenvalue gives the total dissipation of the eigennetwork.

In the vicinity of the frequency at which the networks are $\lambda/4$ long ($\theta_{+1}' = \theta_{-1}' = 180°$), they may be replaced by shunt lumped-element resonators. This allows the variables of the distributed- and lumped-element networks to be related. The circuit considered in this section is the shunt resonator shown in Fig. 3, which applies to the y_{+1} network. It consists of a lumped-element shunt capacitance and

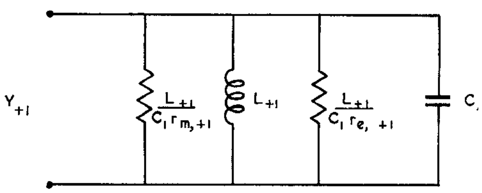

Fig. 3. Approximate lumped equivalent eigennetworks for Y_{+1}.

inductance. The dielectric and magnetic losses are represented by equivalent shunt resistors.

The total normalized input impedance for this circuit is

$$y_{+1} = \frac{Y_{+1}}{Y_0} = j2\delta_{+1}Q_{+1} + Q_{+1}/Q_{u,+1} \tag{29}$$

where

$$2\delta_{+1} \approx 2\left(\frac{\omega - \omega_{+1}}{\omega_{+1}}\right). \tag{30}$$

Here, Q_{+1} and $Q_{u,+1}$ are the loaded and unloaded Q-factors of the equivalent lumped network and ω_{+1} is the resonant frequency of the network.

The input admittance of the distributed network in the vicinity of the frequency at which the network is $\lambda/4$ long is, from (25), given by

$$y_{+1} \approx j\tan\frac{\theta_{+1}}{2} + \alpha_{+1}l\left(1 + \tan^2\frac{\theta_{+1}}{2}\right). \tag{31}$$

Comparing (29) and (31) gives

$$\frac{\theta_{+1}}{2} \approx 2\delta_{+1}Q_{+1} \tag{32}$$

$$\alpha_{+1}l \approx \frac{Q_{+1}}{Q_{u,+1}}\cos^2\frac{\theta_{+1}}{2}. \tag{33}$$

Combining (28) and (33) one now has

$$q_{+1} \approx 4\left(\frac{Q_{+1}}{Q_{u,+1}}\right)\cos^2\frac{\theta_{+1}}{2}. \tag{34}$$

In a similar way one has

$$q_{-1} \approx 4\left(\frac{Q_{-1}}{Q_{u,-1}}\right)\cos^2\frac{\theta_{-1}}{2} \tag{35}$$

$$q_0 = 0. \tag{36}$$

q_0 is zero because it is associated here with a nonresonant eigennetwork.

IV. Scattering Matrix of Lossy Reciprocal 3-Port Junction

For a reciprocal 3-port junction the degenerate eigenvalues are equal:

$$s_{+1} = s_{-1} = s_1 \tag{37}$$

$$q_{+1} = q_{-1} = q_1 \tag{38}$$

$$y_{+1} = y_{-1} = y_1. \tag{39}$$

At the frequency at which maximum power transfer through the junction occurs, the angles of the eigenvalues are

$$\theta_{+1} = \theta_{-1} = \theta_1 = 0. \tag{40}$$

The scattering matrix eigenvalues are therefore

$$s_1 = \left(1 - \frac{q_1}{2}\right) \tag{41}$$

$$s_0 = -1 \tag{42}$$

where

$$q_1 = 4\left(\frac{Q_1}{Q_{u,1}}\right). \tag{43}$$

The coefficients of the scattering matrix are

$$|S_{11}| = \frac{1 - q_1}{3} \tag{44}$$

$$|S_{12}| = |S_{13}| = \frac{2 - q_1/2}{3}. \tag{45}$$

In terms of the original variables one has

$$|S_{11}| = \frac{1}{3} - \frac{4}{3}\left(\frac{Q_1}{Q_{u,1}}\right) \tag{46}$$

$$|S_{12}| = |S_{13}| = \frac{2}{3} - \frac{2}{3}\left(\frac{Q_1}{Q_{u,1}}\right). \tag{47}$$

V. Scattering Matrix of Lossy 3-Port Circulator

For an ideal circulator, the angles of the eigenvalues are

$$\theta_{+1} = \frac{\pi}{3} \tag{48}$$

$$\theta_{-1} = -\frac{\pi}{3}. \tag{49}$$

The eigenvalues of the scattering matrix are, therefore,

$$s_{+1} = \left(1 - \frac{q_{+1}}{2}\right)e^{j\pi/3} \tag{50}$$

$$s_{-1} = \left(1 - \frac{q_{-1}}{2}\right)e^{-j\pi/3} \tag{51}$$

$$s_0 = = -1. \tag{52}$$

Using the above eigenvalues, the entries of the scattering matrix are

$$|S_{11}| = |S_{13}| = \frac{q_{+1}}{6} \tag{53}$$

$$|S_{12}| = 1 - \frac{q_{+1}}{3} \tag{54}$$

provided

$$q_{+1} \approx q_{-1}. \tag{55}$$

In terms of the original variables, the result becomes

$$|S_{11}| = |S_{13}| = \frac{1}{2}\frac{Q_1}{Q_{u,1}} \tag{56}$$

$$|S_{12}| = 1 - \frac{Q_1}{Q_{u,1}}. \tag{57}$$

Hence, $S_{11} \neq 0$, $S_{12} \neq 1$, and $S_{13} = S_{11}$. This is the result obtained in [5], [10] in the case of the lumped-element circulator.

The above scattering coefficients satisfy (1), as can be seen by forming (7).

Fig. 4 shows the relation between S_{11} and S_{12} ($S_{11} = S_{13}$).

VI. Scattering Matrix of Semi-Ideal Circulators

Semi-ideal circulators are ones in which either $S_{11} = 0$, $S_{12} \neq 1$, and $S_{13} \neq 0$, or $S_{11} \neq 0$, $S_{12} \neq 1$, and $S_{13} = 0$. The first situation is obtained when the angle between s_{+1} and s_{-1} is less than $|120|$. The second case is obtained when the angle between the eigenvalues is larger than $|120|$.

The scattering parameters are given in terms of their eigenvalues by

$$3S_{11} = -1 + \left(1 - \frac{q_{+1}}{2}\right)e^{-j\theta_{+1}} + \left(1 - \frac{q_{-1}}{2}\right)e^{j\theta_{+1}} \tag{58}$$

$$3S_{12} = -1 + \left(1 - \frac{q_{+1}}{2}\right)e^{-j(\theta_{+1}-120)} + \left(1 - \frac{q_{-1}}{2}\right)e^{j(\theta_{+1}-120)} \tag{59}$$

$$3S_{13} = -1 + \left(1 - \frac{q_{+1}}{2}\right)e^{-j(\theta_{+1}+120)} - \left(1 - \frac{q_{-1}}{2}\right)e^{j(\theta_{+1}+120)} \tag{60}$$

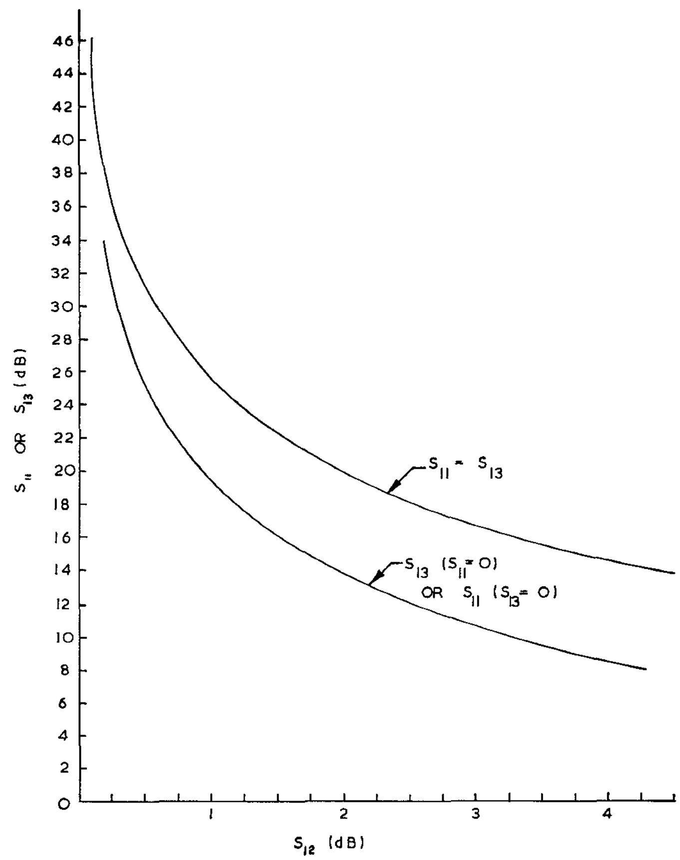

Fig. 4. Relation between S parameters of semi-ideal circulators.

where it has been assumed that one has symmetrical splitting:

$$\theta_{-1} = -\theta_{+1}. \quad (61)$$

In semi-ideal circulators, S_{11} and S_{13} are completely determined by S_{12}, provided it is assumed that the amplitudes of s_{+1} and s_{-1} are equal and the splitting is symmetrical. This means that the latter quantity can be obtained simply by measuring either S_{11} or S_{13}.

The first case to be considered is the one in which the angle between s_{+1} and s_{-1} is such that $S_{11}=0$. This condition is obtained by setting $S_{11}=0$ in (58). The relation between S_{12} and S_{13} is given graphically in Fig. 4.

The second case to be considered here is the one in which the angle between s_{+1} and s_{-1} is such that $S_{13}=0$. This condition is obtained by setting $S_{13}=0$ in (60). The relation between S_{11} and S_{12} is shown in Fig. 4.

VII. Conclusions

The relation between the dissipation and scattering eigenvalues in lossy junctions has been given. The results have been used to directly construct the scattering matrices of a number of 3-port lossy junctions. These results can also be applied to junctions with unequal dissipation eigenvalues that lead to asymmetric frequency responses of the scattering parameters.

Acknowledgment

The author would like to thank Dr. Y. Konishi for some helpful comments.

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Correction to "Scattering by a Ferrimagnetic Circular Cylinder in a Rectangular Waveguide"

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In the above paper,[1] $\sin^{-1} t$ in (17) (Section III-B, p. 524) should be interpreted as $\pi - \mathrm{Sin}^{-1} t$, where $\mathrm{Sin}^{-1} t$ denotes the principal value of $\sin^{-1} t$. Therefore, (23) and (24) should read as follows:

$$S_{11} = \sum_{n=-\infty}^{\infty} (-1)^n A_n \frac{4}{\beta_1 a} \sin\left(\frac{\pi x_0}{a} - n \,\mathrm{Sin}^{-1} \frac{\pi}{k_0 a}\right) \quad (1)$$

and

$$S_{21} = 1 + \sum_{n=-\infty}^{\infty} A_n \frac{4}{\beta_1 a} \sin\left(\frac{\pi x_0}{a} + n \,\mathrm{Sin}^{-1} \frac{\pi}{k_0 a}\right) \quad (2)$$

respectively. Accordingly, Table I should be modified as shown.

The corrected numerical evaluation of $|S_{11}|^2 + |S_{21}|^2$ shows that the unitary condition of the S-matrix is satisfied within a roundoff error for any value of parameters. This is due to the fact that the unitary conditions of the S-matrix is guaranteed for any size of truncation in our formulation of the paper [1]. The following is a proof of this property. The electric field outside the post is expressed as follows:

$$E_z = E_0 \sum_{n=-\infty}^{\infty} J_n(\rho_0^+) e^{-jn\theta_0+} \sin\left(\frac{\pi x_0}{a} + n\alpha\right) + E_0 \sum_{n=-\infty}^{\infty} A_n \sum_{s=-\infty}^{\infty} \left[H_n^{(2)}(\rho_s^+) e^{-jn\theta_s+} - (-1)^n H_n^{(2)}(\rho_s^-) e^{jn\theta_s-}\right]. \quad (3)$$

Consider a region enclosed by two contours $ABCDA$ and F, as shown in Fig. 1. Application of the two-dimensional Poynting theorem to this region yields

$$\mathrm{Re}\left[\frac{1}{2}\int_{AB} (-E_z H_x^*) dx + \frac{1}{2}\int_{DC} E_z H_x^* dx\right] + \mathrm{Re}\left[\frac{1}{2}\oint_F E_z H_\theta^* dl\right] = 0. \quad (4)$$

On the contour AB and DC far from the post, E_z in (3) can be rewritten in the form

$$E_z = E_0 \sin\frac{\pi x}{a} \cdot (e^{-j\beta y} + S_{11} e^{j\beta y}) \quad (5)$$

and

$$E_z = E_0 \sin\frac{\pi x}{a} \cdot S_{21} e^{-j\beta y} \quad (6)$$

respectively. Substitution of (5) and (6) and their corresponding

Manuscript received June 12, 1972.
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